

BERNSTEIN CENTER OF SUPERCUSPIDAL BLOCKS

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ABSTRACT. Let \mathbf{G} be a tamely ramified connected reductive group defined over a non-archimedean local field k . We show that the Bernstein center of a tame supercuspidal block of $\mathbf{G}(k)$ is isomorphic to the Bernstein center of a depth zero supercuspidal block of $\mathbf{G}^0(k)$ for some twisted Levi subgroup of \mathbf{G}^0 of \mathbf{G} .

1. INTRODUCTION

Let \mathbf{G} be a connected reductive group defined over a non archimedean local field k . Assume that \mathbf{G} splits over a tamely ramified extension k^t of k . We will denote the group of k -rational points of \mathbf{G} by G and likewise for other algebraic groups. In [8], Jiu-Kang Yu gives a very general construction of a class of supercuspidal representations of G which he calls *tame*. A tame supercuspidal representation $\pi = \pi_\Sigma$ of G is constructed out of a depth zero supercuspidal representation π_0 of G^0 and some additional data, where \mathbf{G}^0 is a *twisted* Levi subgroup of \mathbf{G} . By twisted, we mean that $\mathbf{G}^0 \otimes k^t$ is a Levi factor of a parabolic subgroup of $\mathbf{G} \otimes k^t$. The additional data, together with \mathbf{G}^0 and π_0 is what we are denoting by Σ in the notation π_Σ . In [4], Kim showed that under certain hypothesis, which are met for instance when the residue characteristic is large, these tame supercuspidals exhaust all the supercuspidals of G .

The depth zero supercuspidal π_0 of G^0 is compactly induced from (K^0, ϱ_0) where K^0 is a compact mod center open subgroup of G^0 and ϱ_0 is a representation of K^0 . The constructed representation π_Σ is compactly induced from (K, ϱ) , where K is a compact mod center open subgroup of G containing K^0 and ϱ is a representation of K . The representation ϱ is of the form $\varrho_0 \otimes \kappa$, where ϱ_0 is seen as a representation of K by extending from K^0 “trivially” (see [8, Sec. 4]) and κ is a representation of K constructed out of the part of Σ which is independent of ϱ_0 .

Let \mathfrak{Z}^π (resp. $\mathfrak{Z}_0^{\pi_0}$) denote the *Bernstein center* of the *Bernstein block* (see Section 4 for these terms) of G (resp. G^0) containing π (resp. π_0). Under certain hypothesis $C(\vec{\mathbf{G}})$ [3, Page 47], we show that:

Theorem. $\mathfrak{Z}^\pi \cong \mathfrak{Z}_0^{\pi_0}$. *Thus, the Bernstein center of a tame supercuspidal block of G is isomorphic to the Bernstein center of a depth zero supercuspidal block of a twisted Levi subgroup of G .*

Let $\mathcal{H}(G, {}^\circ\varrho)$ (resp. $\mathcal{H}(G^0, {}^\circ\varrho_0)$) denote the Hecke algebra of the type constructed out of (K, ϱ) (resp. (K^0, ϱ_0)) (see Sec. 4.2). As a consequence of the above theorem, we obtain

$$Z(\mathcal{H}(G, {}^\circ\varrho)) \cong Z(\mathcal{H}(G^0, {}^\circ\varrho_0)),$$

where $Z(\mathcal{H}(G, {}^\circ\varrho))$ (resp. $Z(\mathcal{H}(G^0, {}^\circ\varrho_0))$) denotes the center of $\mathcal{H}(G, {}^\circ\varrho)$ (resp. $\mathcal{H}(G^0, {}^\circ\varrho_0)$).

In [8, Conj. 0.2], Yu conjectures that $\mathcal{H}(G, {}^\circ\varrho) \cong \mathcal{H}(G^0, {}^\circ\varrho_0)$. This is a special case of his more general conjecture [8, Conj. 17.7]. Assuming certain conditions on π_Σ ([1, Sec. 5.5]) which are satisfied quite often, for instance whenever π_Σ is generic, in Theorem 10 we show that

$$\mathcal{H}(G, {}^\circ\varrho) \cong \mathcal{H}(G^0, {}^\circ\varrho_0).$$

2. NOTATIONS

Throughout this article, k denotes a non-archimedean local field. For an algebraic group \mathbf{G} defined over k , we will denote its k -rational points by G . We will follow standard abuses of notation and terminology and refer, for example, to parabolic subgroups of G in place of k -points of k -parabolic subgroups of \mathbf{G} . Center of \mathbf{G} will be denoted by $\mathbf{Z}_{\mathbf{G}}$. The category of smooth representations of G will be denoted by $\mathfrak{R}(G)$. If K is a subgroup of G and $g \in G$, we denote gKg^{-1} by gK . If ρ is a complex representation of K , ${}^g\rho$ denotes the representation $x \mapsto \rho(g^{-1}xg)$ of gK . For $g \in G$, we say that g *intertwines* ρ if the vector space $\text{Hom}_{{}^gK \cap K}({}^g\rho, \rho)$ is non-zero.

3. YU'S CONSTRUCTION [8]

Let \mathbf{G} be a connected reductive group defined over a non-archimedean local field k . A twisted k -Levi subgroup \mathbf{G}' of \mathbf{G} is a reductive k -subgroup such that $\mathbf{G}' \otimes_k \bar{k}$ is a Levi subgroup of $\mathbf{G} \otimes_k \bar{k}$. Yu's construction involves the notion of a generic \mathbf{G} -datum. It is a quintuple $\Sigma = (\vec{\mathbf{G}}, y, \vec{r}, \vec{\phi}, \rho)$ satisfying the following:

- (1) $\vec{\mathbf{G}} = (\mathbf{G}^0 \subsetneq \mathbf{G}^1 \subsetneq \dots \subsetneq \mathbf{G}^d = \mathbf{G})$ is a tamely ramified twisted Levi sequence such that $\mathbf{Z}_{\mathbf{G}^0}/\mathbf{Z}_{\mathbf{G}}$ is anisotropic.
- (2) y is a point in the extended Bruhat-Tits building of \mathbf{G}^0 over k .
- (3) $\vec{r} = (r_0, r_1, \dots, r_{d-1}, r_d)$ is a sequence of positive real numbers with $0 < r_0 < \dots < r_{d-2} < r_{d-1} \leq r_d$ if $d > 0$, $0 \leq r_0$ if $d = 0$.
- (4) $\vec{\phi} = (\phi_0, \dots, \phi_d)$ is a sequence of quasi-characters, where ϕ_i is a G^{i+1} -generic quasi-character [8, Sec. 9] of G^i ; ϕ_i is trivial on G_{y, r_i}^i , but nontrivial on G_{y, r_i}^i for $0 \leq i \leq d-1$. If $r_{d-1} < r_d$, ϕ_d is nontrivial on G_{y, r_d}^i and trivial on G_{y, r_d}^d . Otherwise, $\phi_d = 1$. Here $G_{y, r}^i$ denote the filtration subgroups of the parahoric at y defined by Moy-Prasad (see [6, Sec. 2.6]).

- (5) ρ is an irreducible representation of $G_{[y]}^0$, the stabilizer in G^0 of the image $[y]$ of y in the reduced building of \mathbf{G}^0 , such that $\rho|_{G_{y,0+}^0}$ is isotrivial and $c\text{-Ind}_{G_{[y]}^0} \rho$ is irreducible and supercuspidal.

Let $K^0 = G_{[y]}^0$, $K^{0+} = G_{y,0+}^0$, $K^i = G_{[y]}^0 G_{y,s_0}^1 \cdots G_{y,s_{i-1}}^i$ and $K^{i+} = G_{[y]}^0 G_{y,s_0+}^1 \cdots G_{y,s_{i-1}+}^i$ where $s_j = r_j/2$ for $i = 1, \dots, d$. In [8, Sec. 11], Yu constructs certain representation κ of $K^d = K^d(\Sigma)$ which is independent of ρ and constructed only out of $(\vec{\mathbf{G}}, y, \vec{r}, \vec{\phi})$. He defines certain subgroups $J^i = (\mathbf{G}^{i-1}, \mathbf{G}^i)(k)_{y,(r_{i-1}, s_{i-1})}$ and $J^{i+} = (\mathbf{G}^{i-1}, \mathbf{G}^i)(k)_{y,(r_{i-1}, s_{i-1}+)}$ for $1 \leq i \leq d$ (see [8, Sec. 3] for the meaning of notations used here). For these groups, one has

$$K^{i-1} J^i = K^i, \quad K^{(i-1)+} J^{i+} = K^{i+}.$$

Also, $K^{i-1} \cap J^i \subset K^{(i-1)+}$. Since ρ is iso-trivial on K^{0+} , one can successively inflate the representation ρ of K^0 to a representation of K^d , which we again denote by ρ , via the maps

$$K^i \twoheadrightarrow K^{i-1} J^i / J^i = K^{i-1} / (K^{i-1} \cap J^i)$$

(see [8, Sec. 4] for details). Write $\rho_\Sigma := \rho \otimes \kappa$.

Theorem 1 (Yu). $\pi_\Sigma := c\text{-Ind}_{K^d}^G \rho_\Sigma$ is irreducible and thus supercuspidal.

The following theorem of Kim [4] says that under certain hypothesis (which are met for instance when the residue characteristic is sufficiently large), the representations π_Σ for various generic \mathbf{G} -datum Σ exhaust all the supercuspidal representations of G .

Theorem 2 (Ju-Lee Kim). *Suppose the hypothesis (Hk), (HB), (HGT) and (HN) in [4] are valid. Then all the supercuspidal representations of G arise through Yu's construction.*

In [3, Theorem 6.6, 6.7] under certain hypothesis denoted by $C(\vec{\mathbf{G}})$ [3, Page 47], Hakim and Murnaghan determine when two supercuspidal representations are equivalent:

Theorem 3 (Hakim-Murnaghan). *Let $\Sigma = (\vec{\mathbf{G}}, y, \vec{r}, \vec{\phi}, \rho)$ and $\Sigma' = (\vec{\mathbf{G}}', y', \vec{r}', \vec{\phi}', \rho')$ be two generic G -data. Set $\phi = \prod_{i=1}^d \phi_i|_{G^0}$, $\phi' = \prod_{i=1}^{d'} \phi'_i|_{G^{0'}}$, $\pi_0 = c\text{-Ind}_{G_{[y]}^0} \rho$ and $\pi'_0 = c\text{-Ind}_{G_{[y'] }^{0'}} \rho'$. Then $\pi_\Sigma \cong \pi_{\Sigma'}$ if and only if there exists $g \in G$ such that $K^d(\Sigma) = {}^g K^{d'}(\Sigma')$ and $\rho_\Sigma = {}^g \rho_{\Sigma'}$ if and only if $G^0 = {}^g G^{0'}$ and $\pi_0 \otimes \phi \cong {}^g (\pi'_0 \otimes \phi')$.*

4. BERNSTEIN CENTER

4.1. Bernstein decomposition. Let $X_k(\mathbf{G}) = \text{Hom}(\mathbf{G}, \mathbb{G}_m)$, the lattice of k -rational characters of \mathbf{G} . Let

$${}^\circ G := \{g \in G : \text{val}_k(\chi(g)) = 0, \forall \chi \in X_k(\mathbf{G})\}.$$

In [5, Section 7], Kottwitz defined a functorial homomorphism $\kappa'_G : G \rightarrow X_*(\mathbf{Z}_{\mathbf{G}})_{I_k}^{\text{Fr}}$. Here $X_*(\mathbf{Z}_{\mathbf{G}})$ denotes the co-character lattice of $\mathbf{Z}_{\mathbf{G}}$, $(\cdot)^{\text{Fr}}$ (resp. $(\cdot)_{I_k}$) denotes taking invariant (resp. coinvariant) with respect to Frobenius Fr (resp. inertia subgroup I_k). The map κ'_G induces a functorial surjective map:

$$(4.1) \quad \kappa_G : G \rightarrow X_*(\mathbf{Z}_{\mathbf{G}})_{I_k}^{\text{Fr}} / \text{torsion}$$

and $\ker(\kappa_G)$ is precisely ${}^\circ G$ (see [2, Sec. 3.3.1]).

Let $X_{\text{nr}}(G) := \text{Hom}(G/{}^\circ G, \mathbb{C}^\times)$ denote the group of *unramified characters* of G . For a smooth representation π of G , the representations $\pi \otimes \chi$, $\chi \in X_{\text{nr}}(G(k))$ are called the *unramified twists* of π .

Consider the collection of all cuspidal pairs (L, σ) consisting of a Levi subgroup L of G and an irreducible cuspidal representation σ of L . Define an equivalence relation \sim on the class of all cuspidal pairs by

$$(L, \sigma) \sim (M, \tau) \text{ if } {}^g L = M \text{ and } {}^g \sigma \cong \tau \nu,$$

for some $g \in G$ and some $\nu \in X_{\text{nr}}(M)$. Write $[L, \sigma]_G$ for the equivalence class of (L, σ) and $\mathfrak{B}(G)$ for the set of all equivalence classes. The set $\mathfrak{B}(G)$ is called the *Bernstein spectrum* of G . We say that a smooth irreducible representation π has *inertial support* $\mathfrak{s} := [L, \sigma]_G$ if π appears as a subquotient of a representation parabolically induced from some element of \mathfrak{s} . Define a full subcategory $\mathfrak{R}(G)^\mathfrak{s}$ of $\mathfrak{R}(G)$ as follows: a smooth representation π belongs to $\mathfrak{R}(G)^\mathfrak{s}$ iff each irreducible subquotient of π has inertial support \mathfrak{s} . The categories $\mathfrak{R}(G)^\mathfrak{s}$, $\mathfrak{s} \in \mathfrak{B}(G)$, are called the *Bernstein Blocks* of G .

Theorem 4 (Bernstein). *We have*

$$\mathfrak{R}(G) = \prod_{\mathfrak{s} \in \mathfrak{B}(G)} \mathfrak{R}(G)^\mathfrak{s}.$$

4.2. Hecke algebra. Let J be a compact open subgroup of G and let (τ, W) be an irreducible representation of J . We call (J, τ) a *compact open datum*. The *Hecke algebra* $\mathcal{H}(G, \tau)$ associated to a compact open datum (J, τ) is the space of compactly supported functions $f : G \rightarrow \text{End}_{\mathbb{C}}(W)$ such that

$$f(j_1 g j_2) = \tau(j_1) f(g) \tau(j_2), \quad j_1, j_2 \in J \text{ and } g \in G.$$

The standard convolution operation gives $\mathcal{H}(G, \tau)$ the structure of an associative \mathbb{C} -algebra with identity.

Let $\mathfrak{R}_\tau(G)$ be the subcategory of $\mathfrak{R}(G)$ consisting of smooth representations which are generated by their τ -isotypic component. If $\mathfrak{R}_\tau(G) = \mathfrak{R}(G)^\mathfrak{s}$ for some $\mathfrak{s} \in \mathfrak{B}(G)$, then we say that (J, τ) is an \mathfrak{s} -type. Let $\mathcal{H}(G, \tau) - \mathfrak{Mod}$ denote the category of non-degenerate $\mathcal{H}(G, \tau)$ modules. If (J, τ) is an \mathfrak{s} -type, then $\mathfrak{R}(G)^\mathfrak{s}$ is equivalent to $\mathcal{H}(G, \tau) - \mathfrak{Mod}$.

4.3. The center of $\mathfrak{R}(G)$. Let \mathcal{C} be an abelian category. The set $\text{End}_{\mathcal{C}}(\text{id})$ of natural transformations of the identity functor of \mathcal{C} is a ring which by definition is the *center* of \mathcal{C} . Denote it by $\mathfrak{Z}(\mathcal{C})$. Explicitly, $z \in \mathfrak{Z}(\mathcal{C})$ is a collection of morphisms $z_A : A \rightarrow A$, one for each object A in \mathcal{C} , such that for any morphism $f : B \rightarrow C$, the diagram

$$\begin{array}{ccc} B & \xrightarrow{f} & C \\ \downarrow z_B & & \downarrow z_C \\ B & \xrightarrow{f} & C \end{array}$$

commutes.

Let R be a ring with identity. Let $\mathfrak{Z}(R)$ (resp. $\mathfrak{Z}(R - \mathfrak{Mod})$) denote the center of R (resp. the center of the category of left R -modules). There is a canonical ring isomorphism

$$(4.2) \quad c \in \mathfrak{Z}(R) \mapsto \mu_c \in \mathfrak{Z}(R - \mathfrak{Mod}),$$

where μ_c acts on each left R -module M by $\mu_c(m) = cm$, for all $m \in M$ (see [7, Sec. 1.6.2]).

Let $\mathfrak{s} \in \mathfrak{B}(G)$. The center of $\mathfrak{Z}(G)$ (resp. $\mathfrak{Z}(G)^\mathfrak{s}$) of the category $\mathfrak{R}(G)$ (resp. $\mathfrak{R}(G)^\mathfrak{s}$) is called the *Bernstein center*. If (J, τ) is an \mathfrak{s} -type, then $\mathfrak{R}(G)^\mathfrak{s} \cong \mathcal{H}(G, \tau) - \mathfrak{Mod}$, and therefore by Equation (4.2), there is a canonical isomorphism

$$(4.3) \quad \mathfrak{Z}(G)^\mathfrak{s} \cong Z(\mathcal{H}(G, \tau)),$$

where $Z(\mathcal{H}(G, \tau))$ denotes the center of $\mathcal{H}(G, \tau)$.

5. SUPERCUSPIDAL BLOCK

Let \mathbf{G} be a connected reductive group over k . Let π be an irreducible supercuspidal representation of G of the form $\pi = c\text{-Ind}_J^G(\tau)$, where J is an open, compact mod center subgroup of G and τ is a representation of J . Write ${}^\circ J = J \cap {}^\circ G$ and let ${}^\circ \tau$ be some irreducible component of $\tau|_{{}^\circ J}$. Then

Proposition 5. [1, Sec. 5.4] *The group ${}^\circ J$ is the unique maximal compact subgroup of J and $({}^\circ J, {}^\circ \tau)$ is a $[G, \pi]_G$ -type in G .*

5.1. Commutativity conditions. Assume that the representation π satisfies the following conditions:

- (1) The representation $\tau|^\circ J$ is irreducible, i.e., ${}^\circ\tau = \tau|^\circ J$.
- (2) Any $g \in G$ which intertwines the representation ${}^\circ\tau$ lies in J .

These conditions are quite frequently satisfied (see [1, Sec. 5.5]), for instance if π admits a Whittaker model ([7, Remark 1.6.1.3]). Under these assumptions, we have:

Proposition 6. [1, Sec. 5.5] *The Hecke algebra $\mathcal{H}(G, {}^\circ\tau)$ associate to the type $({}^\circ J, {}^\circ\tau)$ is commutative.*

6. MAIN RESULT

We use the notations of Section 3. Fix a generic \mathbf{G} -datum $\Sigma = (\vec{\mathbf{G}}, y, \vec{r}, \vec{\phi}, \rho)$. Then $\pi_\Sigma := c\text{-Ind}_{K^d}^G \rho_\Sigma$ is an irreducible supercuspidal representation of G , where ρ_Σ is of the form $\rho \otimes \kappa$ and κ is a representation of K^d , constructed only out of the data $(\vec{\mathbf{G}}, y, \vec{r}, \vec{\phi})$. The representation $\pi_0 = c\text{-Ind}_{G_0^{[y]}}^{G^0} \rho$ of G^0 is depth zero supercuspidal. Set $\mathfrak{s} := [G, \pi_\Sigma]_G$ and $\mathfrak{s}_0 := [G^0, \pi_0]_{G^0}$. Let $\mathfrak{Z}(G)$ (resp. $\mathfrak{Z}(G)^\mathfrak{s}$, resp. $\mathfrak{Z}(G^0)^{\mathfrak{s}_0}$) be the Bernstein center of the category $\mathfrak{R}(G)$ (resp. $\mathfrak{R}(G)^\mathfrak{s}$, resp. $\mathfrak{R}(G^0)^{\mathfrak{s}_0}$). Let $\text{Irr}^\mathfrak{s}(G)$ (resp. $\text{Irr}^{\mathfrak{s}_0}(G^0)$) denote the isomorphism classes of irreducible elements in $\mathfrak{R}(G)^\mathfrak{s}$ (resp. $\mathfrak{R}(G^0)^{\mathfrak{s}_0}$). We assume the hypothesis $C(\vec{\mathbf{G}})$ in [3, Page 47] in the rest of this section.

By functoriality of the map (4.1), the inclusion $\mathbf{G}^0 \hookrightarrow \mathbf{G}$ induces a map

$$(6.1) \quad \chi \in X_{\text{nr}}(G) \mapsto \chi|G^0 \in X_{\text{nr}}(G^0).$$

Theorem 7. *The map $\mathfrak{f}_\Sigma : \pi_\Sigma \otimes \nu \in \text{Irr}^\mathfrak{s}(G) \mapsto \pi_0 \otimes (\nu|G^0) \in \text{Irr}^{\mathfrak{s}_0}(G^0)$, $\nu \in X_{\text{nr}}(G)$, is well defined and is a bijection. Consequently, there is an isomorphism $f_\Sigma : \mathfrak{Z}(G)^\mathfrak{s} \cong \mathfrak{Z}(G^0)^{\mathfrak{s}_0}$.*

Proof. We first prove well definedness. Suppose $\pi_\Sigma \otimes \chi \cong \pi_\Sigma$ for $\chi \in X_{\text{nr}}(G)$. Then we want to show that $\pi_0 \otimes \chi|G^0 \cong \pi_0$. Define a new quintuple $\Sigma_\chi = (\vec{\mathbf{G}}, y, \vec{r}, \vec{\phi}, \rho \otimes (\chi|K^0))$. We have $\pi_\Sigma \otimes \chi \cong c\text{-Ind}_{K^d}^G (\rho \otimes \kappa \otimes (\chi|K^d))$. Since χ is unramified, it follows that $\pi_\Sigma \otimes \chi \cong \pi_{\Sigma_\chi}$. By Theorem 3, $\pi_\Sigma \cong \pi_{\Sigma_\chi}$ is equivalent to (K^d, ρ_Σ) being conjugate to $(K^d, \rho_{\Sigma_\chi})$ by an element $g \in G$. Since $\rho_\Sigma|K^{d+} = \rho_{\Sigma_\chi}|K^{d+}$, it follows that g intertwines $\rho_\Sigma|K^{d+}$. By [8, Prop. 4.4 and 4.1], it implies that $g \in K^d G^0 K^d$. Thus we can assume without loss of generality that $g \in G^0$. Let $\rho' = (\rho \otimes \chi|K^0)$. Then by Theorem 3, we get $\pi_0 \otimes \phi \cong (\pi'_0 \otimes \phi)$ as G^0 -representations, where ϕ is as in Theorem 3 and $\pi'_0 := c\text{-Ind}_{G_0^{[y]}}^{G^0} \rho' \cong \pi_0 \otimes (\chi|G^0)$. It follows that $\pi_0 \otimes \chi|G^0 \cong \pi_0$ and therefore \mathfrak{f}_Σ is well defined. Now if $\chi \in X_{\text{nr}}(G)$ is such that $\pi_0 \otimes \chi|G^0 \cong \pi_0$, then it follows from Theorem 3 or directly that $\pi_{\Sigma_\chi} \cong \pi_\Sigma$, i.e., $\pi_\Sigma \otimes \chi \cong \pi_\Sigma$. This shows that the map \mathfrak{f}_Σ is also injective.

We now prove surjectivity. Now given $\nu \in X_{\text{nr}}(G^0)$, using notation similar to before, write $\Sigma_\nu = (\vec{\mathbf{G}}, y, \vec{r}, \vec{\phi}, \rho \otimes (\nu|K^0))$. Let $({}^\circ K^d, {}^\circ \rho_\Sigma)$ (resp. $({}^\circ K^d, {}^\circ \rho_{\Sigma_\nu})$) be a type constructed out of (K^d, ρ_Σ) (resp. (K^d, ρ_{Σ_ν})) as in Sec. 5. Then ${}^\circ K^0 = G_y^0$ and ${}^\circ K^d = {}^\circ K^0 G_{y, s_0}^1 \cdots G_{y, s_{d-1}}^d$ (see notations in Sec. 3) is the maximal compact subgroup of K^d (see [8, Cor. 15.3]). Since $\rho_\Sigma|{}^\circ K^d = \rho_{\Sigma_\chi}|{}^\circ K^d$, we can assume that $({}^\circ K^d, {}^\circ \rho_\Sigma) = ({}^\circ K^d, {}^\circ \rho_{\Sigma_\nu})$. Now since $({}^\circ K^d, {}^\circ \rho_\Sigma)$ is an \mathfrak{s} -type, it follows that $\pi_{\Sigma_\nu} \cong \pi_\Sigma \otimes \chi$ for some $\chi \in X_{\text{nr}}(G)$. By the argument used in the proof of the well definedness of the map \mathfrak{f}_Σ in the previous paragraph, we get, $\pi_0 \otimes \nu \cong \pi_0 \otimes (\chi|G^0)$, i.e., $\pi_0 \otimes \nu$ is the image of $\pi_\Sigma \otimes \chi$ under \mathfrak{f}_Σ . Thus \mathfrak{f}_Σ is also surjective.

We thus have a bijection $\mathfrak{f}_\Sigma : \pi_\Sigma \otimes \chi \in \text{Irr}^{\mathfrak{s}}(G) \mapsto \pi_0 \otimes (\chi|G^0) \in \text{Irr}^{\mathfrak{s}^\circ}(G^0)$, $\chi \in X_{\text{nr}}(G)$. Since $\mathfrak{Z}(G)^{\mathfrak{s}}$ (resp. $\mathfrak{Z}(G^0)^{\mathfrak{s}_0}$) is canonically the ring of regular functions on $\text{Irr}^{\mathfrak{s}}(G)$ (resp. $\text{Irr}^{\mathfrak{s}^\circ}(G^0)$) [7, Prop. 1.6.4.1], the Theorem follows. \square

For each irreducible object $\tau \in \mathfrak{R}(G)$ and $z \in \mathfrak{Z}(G)$, denote by $\chi_z(\tau)$, the scalar by which z acts on τ .

Corollary 8. *Let $z \in \mathfrak{Z}(G)^{\mathfrak{s}}$ and $\pi \in \text{Irr}^{\mathfrak{s}}(G)$. Then $\chi_z(\pi) = \chi_{\mathfrak{f}_\Sigma(z)}(\mathfrak{f}_\Sigma(\pi))$.*

Proof. This follows from [7, Prop. 1.6.4.1] and Theorem 7. \square

For an algebra \mathcal{A} , denote by $Z(\mathcal{A})$ the center of \mathcal{A} . Let $\mathcal{H}(G, {}^\circ \rho_\Sigma)$ (resp. $\mathcal{H}(G^0, {}^\circ \rho)$) denote the Hecke algebra associated to the compact open data $({}^\circ K^d, {}^\circ \rho_\Sigma)$ (resp. $({}^\circ K^0, {}^\circ \rho)$) (see Sec. 4.2).

Corollary 9. $Z(\mathcal{H}(G, {}^\circ \rho_\Sigma)) \cong Z(\mathcal{H}(G^0, {}^\circ \rho))$.

Proof. Since $({}^\circ K^d, {}^\circ \rho_\Sigma)$ (resp. $({}^\circ K^0, {}^\circ \rho)$) is an \mathfrak{s} -type (resp. \mathfrak{s}_0 -type), this follows from Equation 4.3 and Theorem 7. \square

Now suppose that π_Σ satisfies the commutativity conditions of Sec. 5.1. With this assumption, we get the following result.

Theorem 10. $\mathcal{H}(G, {}^\circ \rho_\Sigma) \cong \mathcal{H}(G^0, {}^\circ \rho)$.

Proof. By assumption, $\rho_\Sigma|{}^\circ K^d$ is irreducible. Since $\rho_\Sigma = \rho \otimes \kappa$ in the notations of Sec. 3, this implies that $\rho|{}^\circ K^0$ is also irreducible. Now by [8, Cor. 15.5], $g \in G^0$ intertwines ${}^\circ \rho$ iff it intertwines ${}^\circ \rho_\Sigma$. But then by assumption, $g \in K^d$. Thus any $g \in G^0$ which intertwines ${}^\circ \rho$ lies in K^0 . This means that π_0 also satisfies the commutativity conditions of Sec. 5.1. Then by Proposition 6, the Hecke algebras $\mathcal{H}(G, {}^\circ \rho_\Sigma)$ and $\mathcal{H}(G^0, {}^\circ \rho)$ are commutative. The Theorem then follows from Corollary 9. \square

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